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Cyclic Quadrilaterals | The Big Picture

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Key words:cyclic, geometry, triangle, segments, point,circle, diameter, circumcircles, intersection, similar, spiral similarity.

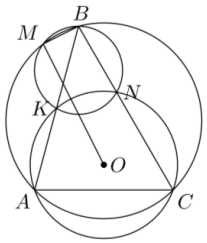
Annotation

This article will fully reveal all the features of Cyclic Quadrilaterals. For example, analyzing the most delicate aspects of the interior and exterior of the Cyclic quadrilaterals are mentioned.

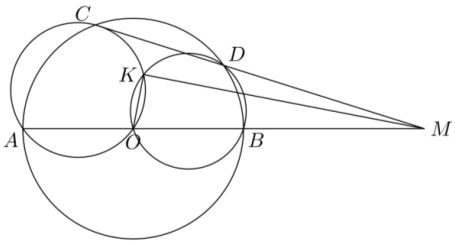
An important skill of an olympiad geometer is being able to recognize known configurations. Indeed, many geometry problems are built on a few common themes. In this lecture, we will explore one such configuration.

1 What Do These Problems Have in Common?

1. (IMO 1985) A circle with center O passes through the vertices A and C of triangle ABC and intersects segments AB and BC again at distinct points K and N, respectively. The circumcircles of triangles ABC and KBN intersects at exactly two distinct points B and M. Prove that $\angle OMB = 90°$.



2. (Russia 1995; Romanian TST 1996; Iran 1997) Consider a circle with diameter AB and center O, and let C and D be two points on this circle. The line CD meets the line AB at a point M satisfying $MB < MA$ and $MD < MC$. Let K be the point of intersection (different from O) of the circumcircles of triangles AOC and DOB. Show that $\angle MKO = 90°$.

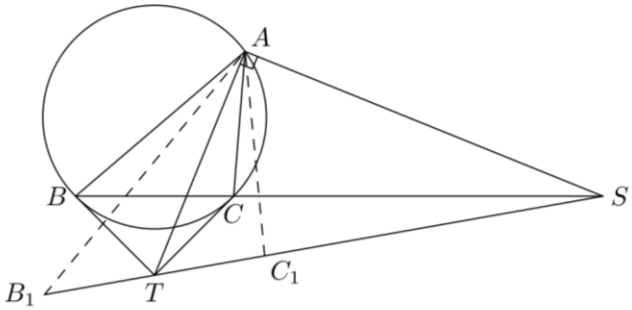


3. (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

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Although these geometric configurations may seem very different at first sight, they are actually very related. In fact, they are all just bits and pieces of one big diagram!

2 One Big Diagram

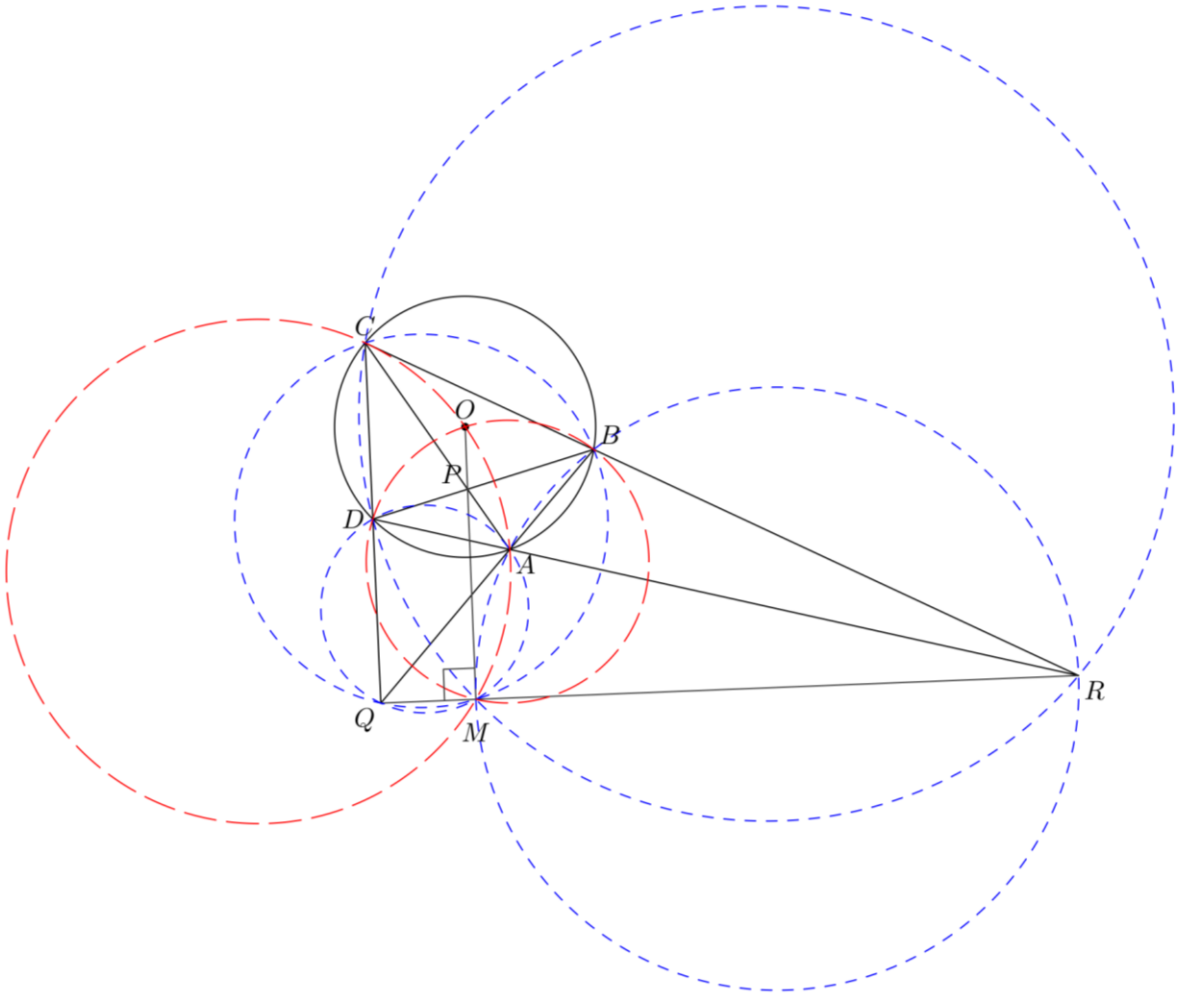


Figure 1: The big picture.

In this lecture, we will try to understand the features of Figure 1. There are a lot of things going on in this diagram, and it can be frightening to look at. Don’t worry, we will go through it

bits and pieces at time. In the process, we will discuss some geometric techniques that are useful in other places as well.

(Can you tell where to find each of the problems in Section 1 in Figure 1? You probably can't at this point, but hopefully you will be able to by the end of this lecture.)

3 Miquel's Theorem and Miquel Point

Fact 1 (Miquel's Theorem)**.** Let ABC be a triangle, and let X, Y, Z be points on lines BC, CA, AB, respectively. Assume that the six points A, B, C, X, Y, Z are all distinct. Then the circumcircles of AYZ, BZX, CXY pass through a common point.

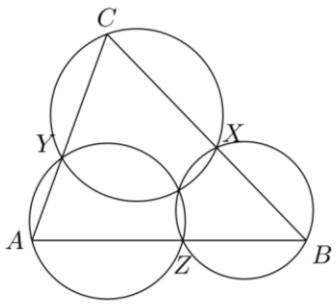


Figure 2: Diagram for Fact 1 (Miquel's Theorem).

Exercise 1. Prove Fact 1. (This is very easy. Just chase[1] a few angles.)

Fact 2 (Miquel point)**.** Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines in the plane, no two parallel. Let $\mathcal{C}_{ijk}$ denote the circumcircle of the triangle formed by the lines ℓ_i, ℓ_j, ℓ_k (these circles are called *Miquel circles*). Then $\mathcal{C}_{123}, \mathcal{C}_{124}, \mathcal{C}_{134}, \mathcal{C}_{234}$ pass through a common point (called the *Miquel point*).

Exercise 2. Prove Fact 2. (Hint: apply Theorem 1)

We want to specialize to the case of a cyclic quadrilateral.

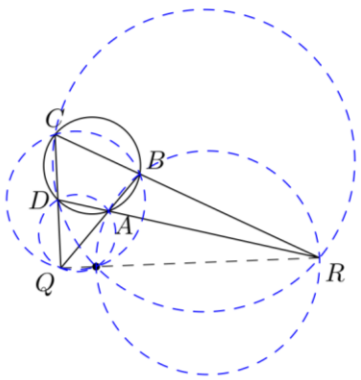


Figure 3: Miquel point for a cyclic quadrilateral $ABCD$.

[1] If you are bothered by configuration and orientation issues (and you should be!), use directed angles.

Fact 3. Let $ABCD$ be a quadrilateral. Let lines AB and CD meet at Q, and lines DA and CB meet at R. Then the Miquel point of $ABCD$ (i.e., the second intersection point of the circumcircles of ADQ and ABR) lies on the line QR if and only if $ABCD$ is cyclic.

Exercise 3. Prove Fact 3. (This is again just easy angle chasing.)

4 An Important Result about Spiral Similarities

A *spiral similarity*[2] about a point O (known as the center of the spiral similarity) is a composition of a rotation and a dilation, both centered at O.

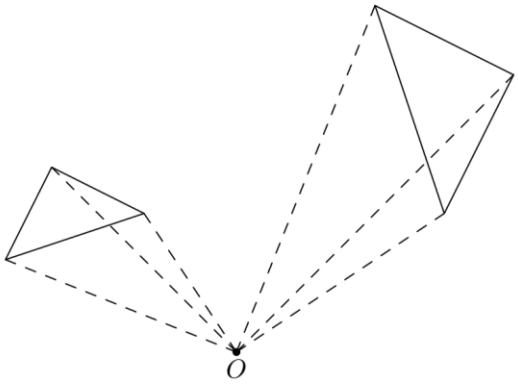


Figure 4: An example of a spiral similarity.

For instance, in the complex plane, if $O = 0$, then spiral similarities are described by multiplication by a nonzero complex number. That is, spiral similarities have the form $z \mapsto \alpha z$, where $\alpha \in \mathbb{C} \setminus \{0\}$. Here $|\alpha|$ is the dilation factor, and $\arg \alpha$ is the angle of rotation. It is easy to deduce from here that if the center of the spiral similarity is some other point, say z_0, then the transformation is given by $z \mapsto z_0 + \alpha(z - z_0)$ (why?).

Fact 4. Let A, B, C, D be four distinct point in the plane, such that $ABCD$ is not a parallelogram. Then there exists a unique spiral similarity that sends A to B, and C to D.

Proof. Let a, b, c, d be the corresponding complex numbers for the points A, B, C, D. We know that a spiral similarity has the form $\mathbf{T}(z) = z_0 + \alpha(z - z_0)$, where z_0 is the center of the spiral similarity, and α is data on the rotation and dilation. So we would like to find α and z_0 such that $\mathbf{T}(a) = c$ and $\mathbf{T}(b) = d$. This amount to solving the system

$$z_0 + \alpha(a - z_0) = c, \qquad z_0 + \alpha(b - z_0) = d.$$

Solving it, we see that the unique solution is

$$\alpha = \frac{c - d}{a - b}, \qquad z_0 = \frac{ad - bc}{a - b - c + d}.$$

Since $ABCD$ is not a parallelogram, we see that $a - b - c + d \neq 0$, so that this is the unique solution to the system. Hence there exists a unique spiral similarity that carries A to B and C to D. □

[2] If you want to impress your friends with your mathematical vocabulary, a spiral similarity is sometimes called a *similitude*, and a dilation is sometimes called a *homothety*. (Actually, they are not quite exactly the same thing, but shhh!)

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